BASIC STEFFENSEN'S METHOD OF HIGHER-ORDER CONVERGENCE

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Abstract: In this paper, we introduce a new analog of a variant of Steffensen's method of fourth-order convergence for solving non-linear equations based on the q-deference operator.

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1. Introduction

Finding the zeros of a nonlinear equation, $f(x) = 0$, is a classical problem of numerical analysis. Analytic methods for solving such equations rarely exit, and therefore, one can hope to obtain only approximate solutions by relying on iteration methods. For a survey of the most important algorithms, some excellent textbooks are available, see [4, 8, 10]. The classical Newtons method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots. \tag{1.1}$$

Being quadratically convergent, Newton's method is probably the best known and most widely used algorithm. Time to time the method has been derived and modified in a variety of ways. One such method derived from Newton's method by approximating the derivative with non-derivative term of difference quotient is Steffensen's method [9, 11]. The method requires two evaluations of function and is quadratically convergent. The interesting iterative scheme is Steffensen's method that has the following form:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{(f(x_n + f(x_n)) - f(x_n))}, \qquad n = 0, 1, 2, \dots. \tag{1.2}$$

In order to control the approximation of the derivative and the stability of the iteration, a Steffensen's type method has been proposed in [2], this approach is based on a better approximation to the derivative $f'(x_n)$ in each iteration. It has the following form:

$$x_{n+1} = x_n - \frac{f(x_n)}{(f(x_n + \alpha_n|f(x_n)|f(x_n)) - f(x_n))/\alpha_n|f(x_n)|f(x_n)}. \tag{1.3}$$

After that, the paper [1] has extended the above result on Banach spaces, obtained its local and semi-local convergence theorems, and made its applications on boundary-value problems by multiple shooting methods.

A family of fourth order methods free from any derivative, satisfying the highest convergence order were established in [12–14].

2. q-Calculus

In the following, q is a positive number, $0 < q < 1$. For $n \in \mathbb{N} = \{0, 1, \ldots\}$, $k \in \mathbb{Z}^+ = \{1, 2, \ldots\}$ and $a,\ a_1, \ldots, a_k \in \mathbb{C}$, the q-shifted factorial, the multiple q-shifted factorial and the $q-$binomial coefficients are defined by

$$(a;q)_0 := 1,\ \ (a;q)_n := \prod_{j=0}^{n-1}(1-aq^j),\ \ (a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j;q)_n, \tag{2.1}$$

and

$$\begin{bmatrix} a \\ 0 \end{bmatrix}_q := 1,\ \text{and}\ \begin{bmatrix} a \\ n \end{bmatrix}_q := \frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-n+1})}{(q;q)_n}, \tag{2.2}$$

respectively. The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$. Moreover $(a;q)_n$ has the representation, cf. [5],

$$(a;q)_n = \sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}a^k. \tag{2.3}$$

The $q-$Gamma function, [5,6], is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z \in \mathbb{C},\ |q| < 1, \tag{2.4}$$

where we take the principal values of q^z and $(1-q)^{1-z}$. In particular

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subseteq \mathbb{C}$ is called a $\mu-$geometric set if for $x \in A$, $\mu x \in A$. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{C}$. The q-difference operator is defined by the formula

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A - \{0\}. \tag{2.5}$$

If $0 \in A$, we say that f has q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n},\ x \in A \tag{2.6}$$

exists and does not depend on x. We then denote this limit by $D_q f(0)$. The q-integration of F. H. Jackson [7] is defined for a function f defined on a q-geometric set A to be

$$\int_a^b f(t)\,d_qt := \int_0^b f(t)\,d_qt - \int_0^a f(t)\,d_qt,\ a, b \in A, \tag{2.7}$$

where

$$\int_0^x f(t)\,d_qt := \sum_{n=0}^{\infty} xq^n(1-q)f(xq^n), \quad x \in A, \tag{2.8}$$

provided that the series converges. A function f which is defined on a q-geometric set A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0), \quad \text{for every } x \in A.$$

The rule of $q-$integration by parts is

$$\int_0^a g(x)D_qf(x)\,d_qx = (fg)(a) - \lim_{n\to\infty}(fg)(aq^n) - \int_0^a D_qg(x)f(qx)\,d_qx. \tag{2.9}$$

If f, g are $q-$regular at zero, the $\lim_{n\to\infty}(fg)(aq^n)$ on the right hand side of (2.9) will be replaced by $(fg)(0)$. The two variable polynomial $\varphi_n(x,a)$, $x, a \in \mathbb{C}$, are defined to be

$$\varphi_0(x,a) := 1, \quad \varphi_n(x,a) := \begin{cases} x^n\,(a/x;q)_n\,, & x \neq 0, \\ (-1)^n q^{\frac{n(n-1)}{2}} a^n, & x = 0. \end{cases} \tag{2.10}$$

In [3], Annaby and Mansour gave q-Taylor series in the following forms

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)}\varphi_k(x,a) + \frac{1}{\Gamma_q(n)}\int_a^x \varphi_{n-1}(x,qt)D_q^n f(t)\,d_qt. \tag{2.11}$$

$$\begin{aligned} f(x) = \sum_{k=0}^{n-1}(-1)^k q^{-\frac{k(k-1)}{2}} \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)}\varphi_k(a,x) \\ + \frac{1}{\Gamma_q(n)}\int_{aq^{-n+1}}^x \varphi_{n-1}(x,qt)D_q^n f(t)\,d_qt, \end{aligned} \tag{2.12}$$

3. A q-Steffensen-secant method

In the following we set $e_n = x_n - a$, $e_n^* = y_n - a$, $z_n = x_n + qf(x_n)$, $y_n = x_n - f(x_n)/f[x_n, z_n]$, where $f[a,b] = \frac{f(a)-f(b)}{a-b}$,

$$A = \frac{D_qf(a)}{\Gamma_q(2)} + \frac{a(1-q)D_q^2f(a)}{\Gamma_q(3)} + \frac{a^2(1-q)^2(1+q)D_q^3f(a)}{\Gamma_q(4)}, \tag{3.1}$$

$$B = \frac{D_q^2f(a)}{\Gamma_q(3)} + \frac{a(1-q)(2+q)D_q^3f(a)}{\Gamma_q(4)}, \tag{3.2}$$

and

$$C = \frac{D_q^3f(a)}{\Gamma_q(4)}. \tag{3.3}$$

Now, we state and prove our q-Steffensen-secant Theorem with fourth order convergence.

Theorem 3.1. *Let $f : \mathcal{D} \to \mathbb{R}$ be a real-valued function with a root $a \in \mathcal{D}$, $\mathcal{D} \subset \mathbb{R}$, and let x_0 be closed enough to a. If $D_q^k(x)$, $k = 1,2,3$ exist, and $D_q(a) \neq 0$, then*

$$x_{n+1} = y_n - \frac{f[x_n,y_n] - f[z_n,y_n] + f[z_n,x_n]}{f^2[x_n,y_n]}f(y_n), \quad n \in \mathbb{N}, \tag{3.4}$$

is fourth-order convergent, and satisfies the following error equation

$$e_{n+1} = A^{-1}B(1+qA)\Big[A^{-1}C(1+qA) - A^{-2}B(3+2qA+2q^2A^2)\Big]e_n^4 + \mathrm{O}(e_n^5), \quad n \in \mathbb{N}. \tag{3.5}$$

Proof. Using the Taylor expansion in (2.11), we have

$$\begin{gathered} f(x_n) = \\ \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a) + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a)(x_n - qa) + \\ \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a)(x_n - qa)(x_n - q^2 a) + \frac{1}{\Gamma_q(4)} \int_a^{x_n} \varphi_3(a, qt) D_q^4 f(t) d_q t. \end{gathered} \tag{3.6}$$

Rearranging the above equation again gives:

$$f(x_n) = Ae_n + Be_n^2 + Ce_n^3 + \mathrm{O}(e_n^4), \tag{3.7}$$

that is

$$\begin{aligned} & f(z_n) = f(x_n + qf(x_n)) = \\ & \frac{1}{\Gamma_q(4)} \int_a^{x_n + qf(x_n)} \varphi_3(a, qt) D_q^4 f(t) d_q t + \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a + qf(x_n)) \\ & + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n)) + \\ & \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n))(x_n - q^2 a + qf(x_n)) \\ & = \mathrm{O}(e_n^4) + \frac{D_q f(a)}{\Gamma_q(2)}(e_n + qf(x_n)) \\ & + \frac{D_q^2 f(a)}{\Gamma_q(3)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1-q)) + \\ & \frac{D_q^3 f(a)}{\Gamma_q(4)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1-q))(e_n + qf(x_n) + a(1-q^2)) \\ & = A(e_n + qf(x_n)) + B(e_n + qf(x_n))^2 + C(e_n + qf(x_n))^3 + \mathrm{O}(e_n^4). \end{aligned} \tag{3.8}$$

Thus,

$$\begin{gathered} f(z_n) = \\ A[1 + qA]e_n + B[1 + 3qA + q^2 A^2]e_n^2 + \\ \Big[C[1 + 4qA + 3q^2 A^2 + q^3 A^3] + 2qB^2[1 + qA]\Big] e_n^3 + \mathrm{O}(e_n^4). \end{gathered} \tag{3.9}$$

Moreover,

$$\begin{aligned} f[z_n, x_n] = & \frac{f(x_n + qf(x_n)) - f(x_n)}{qf(x_n)} \\ = & A + B[2 + qA]e_n + \Big[C[3 + 3qA + q^2 A^2] + qB^2\Big] e_n^2 + \mathrm{O}(e_n^3). \end{aligned} \tag{3.10}$$

Therefore,

$$\begin{gathered} g(x_n) := \frac{f(x_n)}{f[z_n,x_n]} = \\ \mathrm{O}(e_n^4) + e_n - A^{-1}B[1+qA]e_n^2 + \\ \left[A^{-2}B^2[1+qA][2+qA] - qA^{-1}B^2 - A^{-1}C[2+3qA+q^2A^2]\right]e_n^3. \end{gathered} \tag{3.11}$$

Consequently,

$$\begin{gathered} f(y_n) = f(x_n - g(x_n)) = \\ \frac{D_qf(a)}{\Gamma_q(2)}(x_n - a - g(x_n)) + \frac{D_q^2f(a)}{\Gamma_q(3)}(x_n - a - g(x_n))(x_n - qa - g(x_n)) \\ + \frac{D_q^3f(a)}{\Gamma_q(4)}(x_n - a - g(x_n))(x_n - qa - g(x_n))(x_n - q^2a - g(x_n)) \\ + \frac{1}{\Gamma_q(4)}\int_a^{x_n - g(x_n)} \varphi_3(a,qt)D_q^4f(t)d_qt \\ = \mathrm{O}(e_n^4) + \frac{D_qf(a)}{\Gamma_q(2)}(e_n - g(x_n)) + \\ \frac{D_q^2f(a)}{\Gamma_q(3)}(e_n - g(x_n))(e_n + qf(x_n) + a(1-q)) + \\ \frac{D_q^3f(a)}{\Gamma_q(4)}(e_n - g(x_n))(e_n - g(x_n) + a(1-q))(e_n - g(x_n) + a(1-q^2)) \\ = A(e_n - g(x_n)) + B(e_n - g(x_n))^2 + C(e_n - g(x_n))^3 + \mathrm{O}(e_n^4). \end{gathered} \tag{3.12}$$

This means

$$\begin{gathered} f(y_n) = \mathrm{O}(e_n^4) + B[1+qA]e_n^2 - \\ \left[A^{-1}B^2[1+qA][2+qA] - qB^2 - C[2+3qA+q^2A^2]\right]e_n^3, \end{gathered} \tag{3.13}$$

and

$$\begin{gathered} e_n^* = \mathrm{O}(e_n^4) + A^{-1}B[1+qA]e_n^2 - \\ \left[A^{-2}B^2[1+qA][2+qA] - qA^{-1}B^2 - A^{-1}C[2+3qA+q^2A^2]\right]e_n^3. \end{gathered} \tag{3.14}$$

On the other hand

$$\begin{aligned} f[x_n,y_n] =& \frac{f(x_n) - f(y_n)}{g(x_n)} \\ =& A + Be_n + \left[C + A^{-1}B^2[1+qA]\right]e_n^2 + \mathrm{O}(e_n^3). \end{aligned} \tag{3.15}$$

Hence

$$\begin{gathered} f^2[x_n,y_n] = \mathrm{O}(e_n^4) + \\ A^2 + 2ABe_n + \left[2AC + B^2[3+2qA]\right]e_n^2 + \left[2BC + 2A^{-1}B^3[1+qA]\right]e_n^3. \end{gathered} \tag{3.16}$$

But

$$\begin{aligned} f[z_n, y_n] &= \frac{f(z_n) - f(y_n)}{qf(x_n) + g(x_n)} = \\ A + B(1 + qA)e_n + &\Big[C(1 + qA)^2 + A^{-1}B^2(1 + 4qA + 2qA^2)\Big]e_n^2 + \mathrm{O}(e_n^3). \end{aligned} \tag{3.17}$$

So that

$$\begin{aligned} H(x_n) &= \frac{f[y_n, x_n] - f[z_n, y_n] + f[z_n, x_n]}{f^2[y_n, x_n]} = \\ A^{-1} &+ \Big[A^{-2}C(1 + qA) - A^{-3}B(3 + 2qA + 2q^2A^2)\Big]e_n^2 + \\ \Big[-2A^{-3}BC(2 + qA) &+ A^{-4}B^2(5 + 3qA + 4q^2A^2)\Big]e_n^3 + \mathrm{O}(e_n^4). \end{aligned} \tag{3.18}$$

If we multiply $H(x_n)$ by $f(y_n)$ we get

$$\begin{aligned} H(x_n)f(y_n) &= H(x_n)f[y_n, a]e_n^* = \\ \Big[1 + &\Big[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2)\Big]e_n^2 + \\ \Big[-2A^{-2}BC(2 + qA) &+ A^{-3}B^2(5 + 3qA + 4q^2A^2)\Big]e_n^3 + \mathrm{O}(e_n^4)\Big]e_n^*. \end{aligned} \tag{3.19}$$

Taking in consideration that x_{n+1} is nothing but $y_n - H(x_n)f(y_n)$ we get

$$\begin{aligned} x_{n+1} =& y_n - H(x_n)f(y_n) \\ =& x_n - \Big[1 + \Big[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2)\Big]e_n^2 + \\ &\Big[-2A^{-2}BC(2 + qA) + A^{-3}B^2(5 + 3qA + 4q^2A^2)\Big]e_n^3 + \mathrm{O}(e_n^4)\Big]e_n^*. \end{aligned} \tag{3.20}$$

Thus

$$\begin{aligned} e_{n+1} &= \Big[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2) + \mathrm{O}(e_n)\Big]e_n^2e_n^* \\ &= A^{-1}B[1 + qA]\Big[A^{-1}C(1 + qA) - A^{-2}B(3 + 2qA + 2q^2A^2)\Big]e_n^4 + \mathrm{O}(e_n^5). \end{aligned} \tag{3.21}$$

This completes the proof. □

In order to compare our new method with Steffensen's method, we give the following example.

Example: In this example we take

$$f(x) = \cos(x) - x.$$

The root of $f(x)$ is $a = 0.7390851332$. Then the sequence $\{x_n\}_n$

$$x_{n+1} = y_n$$
$$- \frac{qf^2(x_n)\Big[\Big(\cos(y_n) - x_n\Big)E_q(x_n) + qf^2(x_n)\Big]}{\Big[E_q(x_n) + f(x_n)\Big]\Big[\Big(\cos(y_n) - \cos(x_n)\Big)E_q(x_n) + qf^2(x_n)\Big]}$$
$$+ \frac{qf^4(x_n)\Big[\Big(\cos(y_n) - x_n\Big)E_q(x_n) + qf^2(x_n)\Big]}{\Big[E_q(x_n) + f(x_n)\Big]\Big[\Big(\cos(y_n) - \cos(x_n)\Big)E_q(x_n) + qf^2(x_n)\Big]^2},$$

is fourth-order convergent, where

$$E_q(x_n) = \cos\Big(q\cos(x_n) + (1-q)x_n\Big) - (1+q)\cos(x_n) + qx_n ,$$
$$y_n = x_n - \frac{q\Big(\cos(x_n) - x_n\Big)^2}{E_q(x_n)} .$$

Taking $x_0 = 0$, for $q = 0.5$, we find

	x_1	x_2	x_3	x_4
Our's	0.8617217519	0.7399567610	0.7390851885	0.7390851332
Steffensen's	2.175342650	0.76343368	0.7613122807	0.7595358304

Taking $x_0 = 1.1$, for $q = 0.9$, we find

	x_1	x_2	x_3	x_4
Our's	0.7063822168	0.7388491909	0.7390851206	.7390851333
Steffensen's	0.8296038833	0.8040964255	0.7902498570	0.7814206993

Taking $x_0 = 1.35$, for $q = 0.001$, we find

	x_1	x_2	x_3	x_4
Our's	0.8144712303	0.74139713209	0.7390873914	0.7390851090
Steffensen's	0.7429374052	0.7428816874	0.7428275625	0.7427749629

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